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NUMERICAL MODELING OF THE SINGLE-PHASE STEFAN PROBLEM IN A LAYER WITH TRANSPARENT AND SEMITRANSPARENT BOUNDARIES

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Numerical modeling of the single-phase Stefan problem in a semitransparent layer with transparent, nonabsorbing, and partially radiation-absorbing boundaries is performed. It is shown that at low temperatures of the medium, convection is a determining factor on the boundary of the irradiated sample, and at high temperatures, radiation is predominant. The absence of absorption on the boundaries of the layer leads to acceleration of the heating of the plate and considerable deceleration of melting processes.

Key words:*radiation, absorption, heating, phase transition, convection, heat conduction.*

Introduction. Interest in studies of radiative–conductive heat transfer in transparent media taking into account a first-order phase transition has considerably increased because of practical applications of this process (glassmaking, crystal growth, thermal protection, and ice melting). The first numerical simulation of the single-phase Stefan problem in a material layer with different parameters of volume absorption and emission of radiation was performed in [1]. In this paper, a transformation was introduced that fixes the phase transition front [2]. This facilitates a qualitative analysis of the solution results but hinders the calculation process. The radiation part of the problem was solved by the method of discrete ordinates and a directional beam. Following [1], the single-phase Stefan problem was considered in [3] taking into account the radiation reflection from the boundaries of the transparent layer. In this study, unlike in [1], the radiation term of the energy equation was determined using the effective mean-flux method [4]. The indicated method yielded results in good agreement with the results of [1].

Formulation of the Problem and Method of Solution. In the present paper, we study the heating and subsequent melting of an infinite plane–parallel sample with a semitransparent (absorbing, emitting, and nonscaterring) gray medium. As in [5], as a fist step, we consider the unsteady radiative–conductive heat transfer in the process of heating of the sample by radiation and convection. In the second step, when boundary of the sample reaches the melting point, the Stefan problem is considered. The liquid phase formed on the boundary is sublimated and carried away by convection. The position of the interface $S(t)$ is determined by solving the boundary-value problem, which amounts to determining the temperature fields and fluxes in a solid-phase layer of thickness varying from $x = 0$ to $x = S(t)$ (Fig. 1).

Under the assumption of constant thermal properties of the medium, the energy equation in the solid plate is written as

$$\frac{\partial T(x,t)}{\partial t} = a\,\frac{\partial^2 T(x,t)}{\partial x^2} - \frac{1}{\rho c_p}\,\frac{\partial E}{\partial x}, \qquad 0 \leqslant x \leqslant S(t), \quad t > 0, \tag{1}$$

where a and ρ are the thermal diffusivity and density of the solid phase, c_p is the specific heat at constant pressure, E is the density of the resultant radiation flux in the gray medium layer, which is expressed in terms of the forward radiation intensity $I^+(x,\mu,t)$ and backward radiation intensity $I^-(x,\mu,t)$ as follows:

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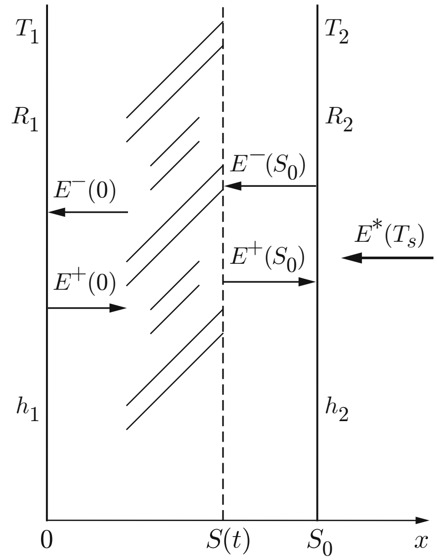

Fig. 1. Geometrical diagram of the problem.

$$E(x,t) = 2\pi \int_0^1 (I^+(x,\mu,t) - I^+(x,\mu,t))\mu\, d\mu = E^+(x,t) - E^-(x,t).$$

Here $E^{\pm}(x,t)$ is the flux density of the hemispherical (within the solid angles $\Omega = \pm 2\pi$) incident radiation (see Fig. 1).

For the case of transparent (nonabsorbing) boundaries in the problem considered in the second step of the solution taking into account the Stefan condition on the interface, the boundary conditions are written as

$$-\lambda \frac{\partial T(0,t)}{\partial x} = h_1(T_1 - T(0,t)); \tag{2}$$

$$\lambda \frac{\partial T(S(t),t)}{\partial x} - h_2(T_2 - T(S(t),t)) = \rho\gamma \frac{\partial S(t)}{\partial t}. \tag{3}$$

System (1)–(3) is supplemented by the initial condition

$$T(x,0) = f(x), \qquad S(0) = S_0. \tag{4}$$

Here λ is the thermal conductivity of the sample, h_i is the coefficient of heat transfer to the ambient medium, T_i is the temperature of the medium on the left and right of the sample, and γ is the latent heat of melting; the subscripts $i = 1$ and 2 correspond to the left and right boundaries of the sample, respectively.

In the first step of the solution, $S(t)$ in the energy equation (1) should be set identically equal to the initial thickness S_0 and the right side of Eq. (3) should be set equal to zero.

Rendering the energy equation dimensionless, we use the transformation of [2], which transforms the phase transition front to a fixed boundary for $\xi = x/S(t) \equiv 1$. This is done using the dimensionless variables $\theta = T/T_f$, $\xi = x/S(t)$, $s(\eta) = S(t)/S_0$, and $\eta = \lambda t/(\rho c_p S_0^2)$. Equation (1) in the indicated variables becomes

$$\frac{\partial\theta(\xi,\eta)}{\partial\eta} = \xi \frac{\dot{s}}{s} \frac{\partial\theta(\xi,\eta)}{\partial\xi} + \frac{1}{s^2} \frac{\partial^2\theta(\xi,\eta)}{\partial\xi^2} - \frac{1}{Ns} \frac{\partial\Phi(\xi,\eta)}{\partial\xi}, \qquad 0 \leqslant \xi \leqslant 1, \tag{5}$$

and boundary conditions (2) and (3) become

$$-\frac{\partial\theta(0,\eta)}{\partial\xi} = s\,\mathrm{Bi}_1(\theta_1 - \theta(0,\eta)); \tag{6}$$

$$\frac{\partial\theta(1,\eta)}{\partial\xi} - s\,\mathrm{Bi}_2(\theta_2 - \theta(1,t)) = \frac{s\dot{s}}{\mathrm{St}}. \tag{7}$$

Here $N = \lambda/(4\sigma_0 T_r^3 S_0)$ is the radiative–conductive parameter, $\Phi^{\pm}(\xi, \tau) = E^{\pm}(x, t)/(4\sigma_0 T_r^4)$ is the dimensionless radiation flux density, $\mathrm{Bi}_i = h_i S_0/\lambda$ is the Biot number, $\mathrm{St} = c_p T_r/\gamma$ is the Stefan number, $T_r = T_f$ is the determining temperature equal to the phase-transition temperature, $\dot{s} = ds/d\eta$, and σ_0 is the Stefan–Boltzmann constant.

The initial condition (4) becomes

$$\theta(\xi, 0) = f(\xi), \qquad s(0) = 1. \tag{8}$$

If the boundaries of the layer partially absorb, reflect, and transmit radiation, the boundary conditions of the problem (2) and (3) are written as

$$-\lambda \frac{\partial T(0, t)}{\partial x} + A_1[E^-(0, t) + \sigma_0 T_1^4] - \varepsilon_1(1 + n^2)\sigma_0 T^4(0, t) = h_1(T_1 - T(0, t)); \tag{9}$$

$$\lambda \frac{\partial T(S(t), t)}{\partial x} - h_2(T_2 - T(S(t), t)) - A_2[E^+(S(t), t) + E^*] + \varepsilon_2(1 + n^2)\sigma_0 T^4(S(t), t) = \rho\gamma \frac{dS(t)}{dt}. \tag{10}$$

Here A_i satisfies the balance relation of the dimensionless fluxes on the boundaries of the sample

$$A_i + R_i + D_i = 1, \tag{11}$$

where A_i, R_i, and D_i are the values of the hemispherical absorption, reflection, and transmission coefficients of the boundaries, respectively, and ε_i is the emissivity factor of the boundaries (below, it is assumed that $\varepsilon_i = A_i$).

The dimensionless form of Eqs. (9) and (10) is written as

$$-\frac{\partial \theta(0, \eta)}{\partial \xi} + s\,\mathrm{Bi}_1(\theta(0, \eta) - \theta_1) - \frac{A_1 s}{N}\Big(\Phi^- + \frac{\theta_1^4}{4} - \frac{1 + n^2}{4}\,\theta^4(0, \eta)\Big) = 0; \tag{12}$$

$$\frac{\partial \theta(1, \eta)}{\partial \xi} - s\,\mathrm{Bi}_2(\theta_2 - \theta(1, \eta)) - \frac{A_2 s}{N}\Big(\Phi^+ + F^* - \frac{1 + n^2}{4}\,\theta^4(1, \eta)\Big) = \frac{s\dot{s}}{\mathrm{St}}. \tag{13}$$

In the formulation considered, solution of the problem reduces to determining the temperature $\theta(\xi, \eta)$ and density of the resultant radiation flux $\Phi(\xi, \eta)$ in the region $G = \{0 \leqslant \xi \leqslant 1;\ 0 \leqslant \eta \leqslant \eta_1\}$, which is a flat layer of the solid phase. The position of the phase transition front $s(\eta)$ varies from 1 to 0.

The boundary-value problems (5)–(7) and Eqs. (5), (12), and (13) are solved by a finite-difference method, and the nonlinear system of implicit difference equations by a sweep method and iterations. In Eqs. (5)–(7), (12), and (13), the radiation fluxes are internal sources and are found by solving the radiation transfer equation with a known temperature distribution for the flat layer of the emitting and absorbing medium.

As applied to radiation heat transfer, the modified mean-flux method [3, 4] offers wide capabilities for calculations of the radiation transfer in absorbing and emitting media taking into account radiation reflection from the boundary surfaces. In this method, the integrodifferential equation of radiation transfer reduces to a system of two non-linear differential equations for the a flat layer of a semitransparent medium. For hemispherical fluxes, the differential analog of the radiation transfer equation is written as

$$\frac{d}{d\tau}\,(\Phi^+(\tau, \eta) - \Phi^-(\tau, \eta)) + (m^+(\tau)\Phi^+(\tau, \eta) - m^-(\tau)\Phi^-(\tau, \eta)) = n^2\Phi_0; \tag{14}$$

$$\frac{d}{d\tau}\,(m^+(\tau)\delta^+(\tau)\Phi^+(\tau, \eta) - m^-(\tau)\delta^-(\tau)\Phi^-(\tau, \eta)) + (\Phi^+(\tau, \eta) - \Phi^-(\tau, \eta)) = 0. \tag{15}$$

The boundary conditions on the transparent, diffusely emitting, and reflecting surfaces are given by

$$\Phi^+(0, \eta) = (1 - R_1)\,\frac{\theta_1^4}{4} + \Big(1 - \frac{1 - R_1}{n^2}\Big)\Phi^-(0, \eta); \tag{16}$$

$$\Phi^-(1, \eta) = (1 - R_2)\Big(F^* + \frac{\theta_s^4}{4}\Big) + \Big(1 - \frac{1 - R_2}{n^2}\Big)\Phi^+(1, \eta). \tag{17}$$

Here $\theta_1^4 = T_1^4/T_r^4$, $\theta_s^4 = T_s^4/T_r^4$, and $F^* = E^*/(4\sigma_0 T_r^4)$ is the dimensionless density of the flux incident on the plate from the right and n is the index of refraction.

The boundary conditions on the diffusely reflecting, transmitting, and partially absorbing (emitting) surfaces are given by

$$\Phi^+(0,\eta) = \varepsilon_1 n^2\,\frac{\theta^4(0,\eta)}{4} + D_1\,\frac{\theta_1^4}{4} + \Big(1 - \frac{1-R_1}{n^2}\Big)\Phi^-(0,\eta); \tag{18}$$

$$\Phi^-(1,\eta) = \varepsilon_2 n^2\,\frac{\theta^4(1,\eta)}{4} + D_2 F^* + \Big(1 - \frac{1-R_2}{n^2}\Big)\Phi^+(1,\eta); \tag{19}$$

$$\Phi^\pm(\tau,\eta) = \frac{2\pi}{4\sigma T_r^4}\int\limits_{0(-1)}^{1(0)} I(\tau,\mu)\mu\,d\mu,$$

$$m^\pm(\tau) = \int\limits_{0(-1)}^{1(0)} I(\tau,\mu)\,d\mu \Big/ \int\limits_{0(-1)}^{1(0)} I(\tau,\mu)\mu\,d\mu,$$

$$\delta^\pm(\tau) = \int\limits_{0(-1)}^{1(0)} I(\tau,\mu)\mu^2\,d\mu \Big/ \int\limits_{0(-1)}^{1(0)} I(\tau,\mu)\mu\,d\mu.$$

Here I is the radiation intensity, μ is the cosine of the angle between the propagation direction of the radiation and the x axis, $\tau = \alpha S(t)$ is the optical thickness of the layer at the time t, α is the absorption coefficient, R_i is the coefficient of hemispherical radiation reflection by the nonabsorbing boundaries of the layer (calculated by the Walsh–Dunkl formula). The values of $m^\pm$ and $\delta^\pm$ are determined from the recursive equation obtained by formal solution of the radiation transfer equation [6, 7]. The radiation flux density is given by

$$\Phi(\tau,\eta) = \Phi^+(\tau,\eta) - \Phi^-(\tau,\eta). \tag{20}$$

The radiation problem is solved using iterations, in each step of which the boundary-value problem (14)–(20) is solved by the matrix factorization method. The rapid convergence of this method of solution ensures high-accuracy results.

The temperature and radiation flux fields were calculated numerically, and the position of the phase transition front and the temperature variation on the left boundary of the sample were determined. The calculations were performed for the following parameter values: $S_0 = 0.1$ m, $T_f = 1000$ K, $T_1 = 300$ K, $T_2 = 900$ K, $E^* = 120$ kW/m^2, $\rho = 2000$ kg/m^3, $\lambda = 1$ W/(m$\cdot$ K), $a = 10^{-6}$ m^2/sec, $\gamma = 500$ kJ/kg, $h_{1,2} = 1$ W/(m$^2\cdot$ K), $n = 1.5$, $\alpha = 10$ m^{-1}, and $A_{1,2} = 0.1$; for the transparent boundaries, the reflection coefficient $R_{1,2} = 0.092$ was calculated by the Walsh–Dunkl formula, and for the semitransparent boundaries, $R_{1,2} = 0.1$.

The optimal parameters of the external action on a volumetrically absorbing layer with transparent (nonabsorbing) boundaries were determined using the results of [5]. This allows uniform heating of the sample to be implemented up to the moment the phase transition conditions are attained. In the present study, we considered situations where the sample was subjected to a convective flow on the right at a fixed temperature $T_2 = 900$ K.

Figure 2 gives the results of calculations of the heating and melting of a plate with reflecting and nonabsorbing boundaries due to radiative–convective heating of the right surface. In this case, the processes occur under the action of the maximum radiation flux penetrating into the plate (the reflection losses by the right boundary are minimal). The nature of the temperature fields in the layer (Fig. 2a) is determined by the radiation and depends slightly on ambient temperature. The plate is rapidly heated to the phase transition temperature on the right boundary (curve 2 in Fig. 2a). In a detailed consideration of the temperature field in the phase transition process (Fig. 2b), one can clearly see the overheating zone near the right boundary of the sample, which is determined by the heat transfer for a fixed value of the phase transition temperature T_f. The maximum overheating is observed for $s = 0.9$ (Fig. 2b). During the melting of the plate, the temperature maximum is shifted to its middle and the temperature distribution in the plate ($s = 0.223$) becomes quasi-isothermal. The melting process is extended in time. The resultant radiation flux of negative magnitude, increases monotonically with distance from the right wall, takes the nature of a linear distribution with time (Fig. 2c and d), and becomes nearly constant in the process of melting

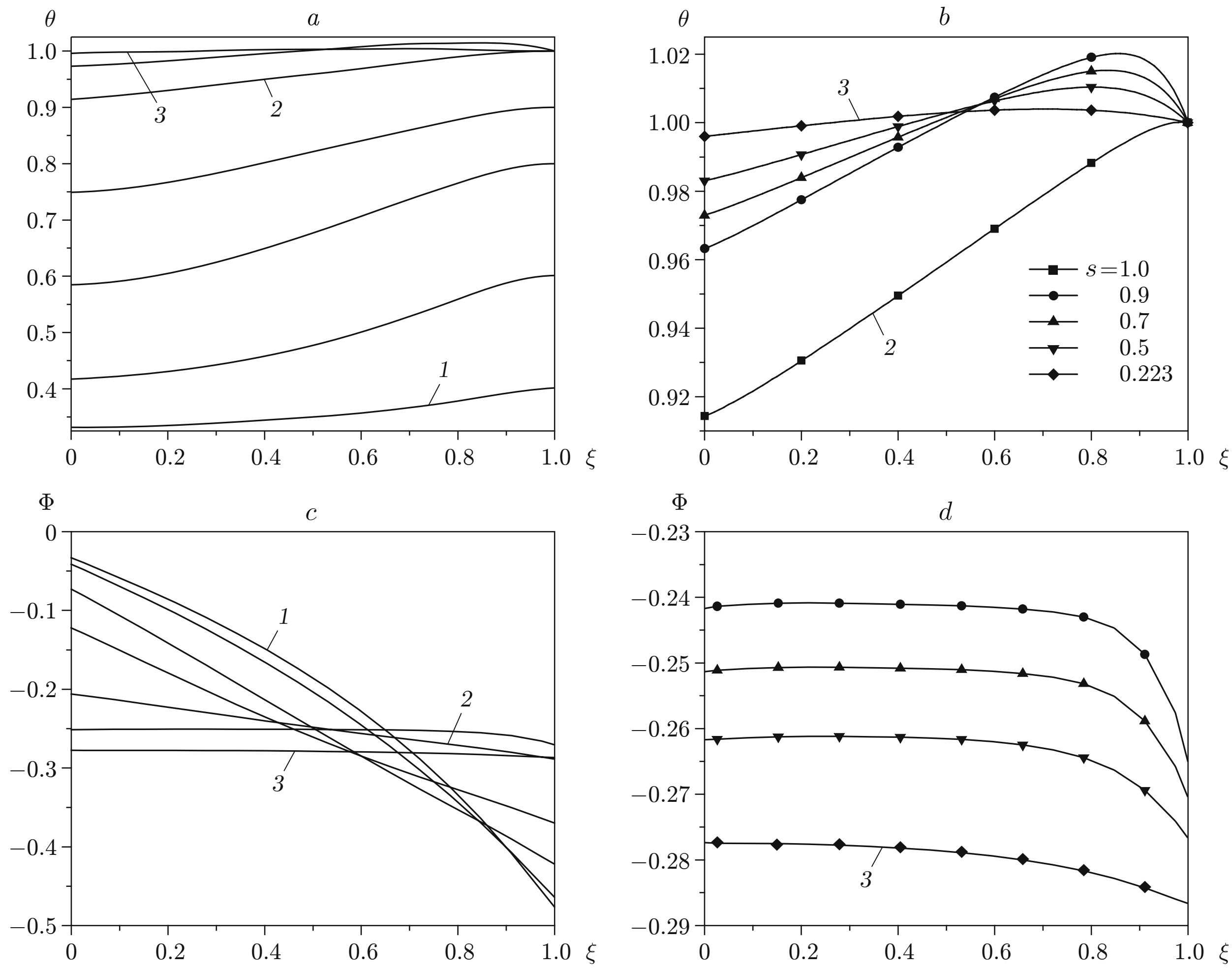

Fig. 2. Temperature distributions (a and b) and radiation flux density distributions (c and d) in the sample in the absence of absorption for $R_2 = 0.092$ and $T_2 = 900$ K: (a and c) heating and melting; (b and d) melting; curves 1 refer to $t = 55$ sec (the beginning of the process), curve 2 refer to $t = 1003$ sec (the onset of the phase transition), and curves 3 refer to $t = 30{,}753$ sec (the end of the process).

and thinning of the plate ($s \to 0.2$) (curves 3 in Fig. 2c and d). The latter is due to the quasi-isothermicity of the material of the thin plate with a small optical thickness. In this case, because of the nearly isothermal temperature distribution, the total flux $q = -\lambda\,\partial T/\partial x + E$ becomes almost equal to the resultant radiation flux density: $q \approx E$.

It should be noted that under the conditions considered, where the boundaries of the layer do not absorb but transmit radiation, the phase transition on the right side of the layer leads to a considerable increase in the time of melting. In this case, the melting process is completed for a dimensionless layer thickness $s \leqslant 0.2$.

Weak absorption of radiation by the boundaries ($A_i = \varepsilon_i = 0.1$) changes the nature of the plate heating (Fig. 3a) and reduces the time of melting. In the neighborhood of the heated boundary, an extremum appears that is due to radiation absorption upon heating (see Fig. 3a) and the occurrence of phase transition with fixed value of the melting point (Fig. 3b). The resultant radiation flux density distribution (Fig. 3c and d) is characterized by the presence of breaks due to optical nonuniformity in the phase transition period (Fig. 3d), and at the end of the process, it becomes quasilinear (curve 3 in Fig. 3d).

From Fig. 4a, one can see that the temperature of the left boundary does not depend on the optical properties of the boundaries, which is explained by convective cooling of the boundary at $T_1 = 300$ K. The considerable temperature growth on the left boundary of the layer is observed during the first 1000 sec of the process, i.e., before the onset of the phase transition, after which the growth curve is stabilized and reaches a quasi-stationary regime.

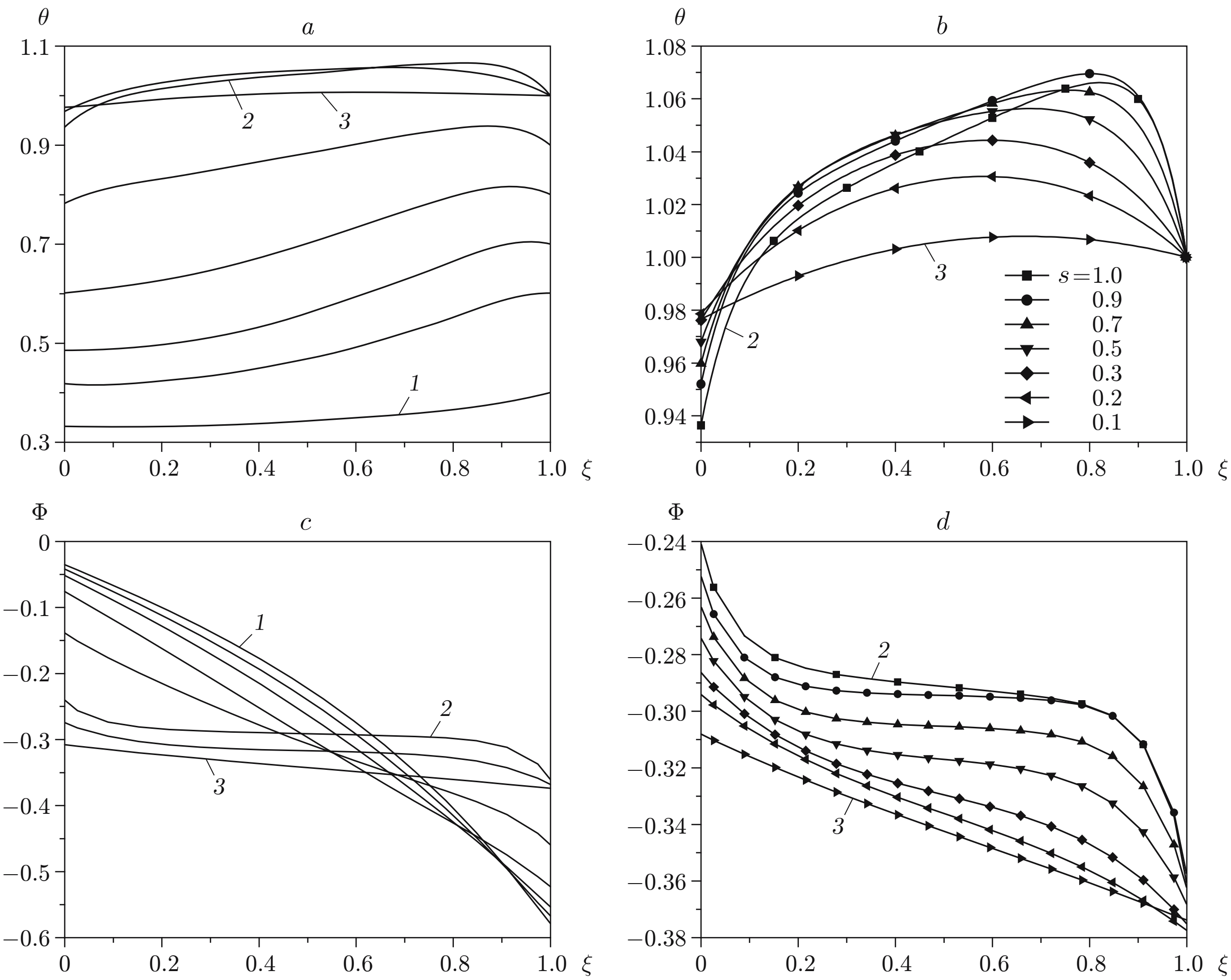

Fig. 3. Temperature distributions (a and b) and radiation flux density distributions (c and d) in the sample for partial radiation absorption ($A_2 = 0.1$) and $R_2 = 0.1$ $T_2 = 900$ K: (a, c) heating and melting; (b, d) melting; curves 1 refer to $t = 127$ sec (the beginning of the process), curves 2 refer to $t = 1292$ sec (the onset of the phase transition), and curves 3 refer to $t = 12{,}890$ sec (the end of the process).

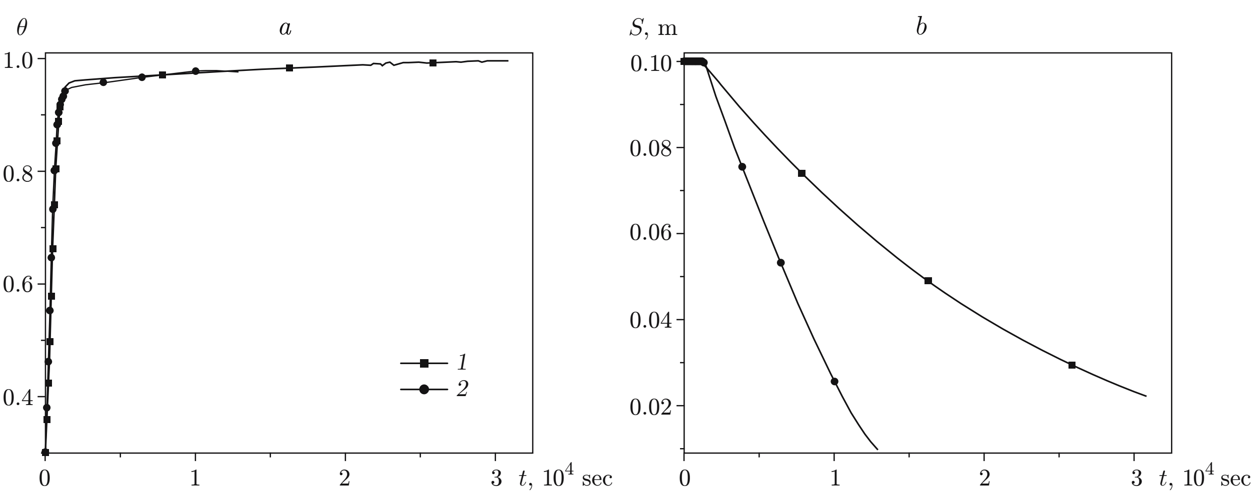

Fig. 4. Time evolution of the temperature of the left boundary (a) and the position of the melting front (b) at $T_2 = 900$ K and $A_2 = 0.1$: 1) transparent boundary; 2) semitransparent boundary.

The position of the melting front during the heating and melting of the plate is shown in Fig. 4b. It is obvious that the time of plate melting depends even on an indignant absorption of the radiation by the right boundary. We note that the indicated circumstance can be of significance when taking account the nebulosity of the surface layer due to a change in the optical properties of the plate material upon the phase transition.

Conclusions. The results obtained in the present study allow one to estimate the role of boundary conditions in the formation of a thermal field in a flat plate during its heating and melting. The absence of radiation absorption by the plate boundaries leads to a considerable deceleration of the melting process compared to the case of total radiation absorption by the sample boundaries considered in [1, 3]. The insignificant absorption on the irradiated boundary of the plate due to a possible change in the optical properties of the plate material upon the phase transition considerably accelerates the processes compared to the case of nonabsorbing boundaries.

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